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Bounded Risk Two-stage Estimation Procedure for a $U(a\theta, b\theta)$ Distribution

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Abstract

In the literature, an extensive work on sequential fixed-width confidence interval for the parameter of $U(\theta, m\theta)$ model, where $m > 1$ is known, is available. In this article, we propose a two-stage sampling procedure for estimating the parameter θ of $U(a\theta, b\theta)$ distribution, where $a < b$ are positive and known. Here, the risk of an estimator $\hat{\theta}$ of θ is less than a pre-assigned number w (>0), that is, $R(\hat{\theta},\theta) = AE_{\theta}[(\hat{\theta}-\theta)^2] \le w$, $0 < A < \infty$ is known. We determine the parameter B_k of stopping variable so that the risk is uniformly bounded by a pre-assigned value w. We have also tabulated the values of the expected stopping time and its standard deviation (SD).

Keywords

Fixed sample size (FSS) procedure, bounded risk estimation, sequential estimation, stopping variable, two-stage sampling

AMS 2000 subject classification: 62L15; 60G40; 62F12; 62F15.

1. Introduction

Graybill and Connell,[1] Cooke,[2,3] Govindarajulu,[4,5] Akahira and Koike,[6] and Koike[7] have introduced many sequential estimation methods for uniform distribution. The problem of obtaining confidence intervals having a specified width for the parameter in the density $U(\theta, m\theta)$ distribution, where $m > 1$ is known and $\theta > 0$, have been considered by Patil and Rattihalli.[8] Bhattacharjee and Mukhopadhyay[9] have discussed the purely sequential procedure for the unknown parameter θ of $U(0, \theta)$ distribution. The unknown parameter θ is estimated by four different estimators in stopping rule, and the two different estimators of θ were proposed in the loss function. Patil[10] has considered the two-stage estimation procedure for the parameter of $U(\theta, m\theta)$ distribution. Bhattacharjee and Mukhopadhyay[11] have proposed the purely sequential minimum risk point estimation procedure for the parameter θ of the $U(0,\theta)$ distribution. Patil[12] has considered the purely sequential procedure for the parameter of the $U(\theta, m\theta)$ distribution.

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Various methods of sequential estimation of the scale parameter of an exponential distribution have been introduced by many authors, for example, Zacks and Mukhopadhyay,[13,14] Mukhopadhyay and Pepe[15], Zacks,[16] etc. Zacks and Khan[17] studied the confidence intervals of the mean and scale parameter of a gamma distribution. Mahmoudi and Roughani[18] have considered bounded risk estimation of the scale parameter of a gamma distribution in a two-stage sampling procedure. For details, see Ghosh et al.[19]

The $U(a\theta, b\theta)$ distribution is appropriate in a following situation. Consider an agriculture experiment where we want to study the impact of unknown soil fertility gradient θ of a plot on the yield/growth of a certain crop, which is an observable random variable, say X, whose range depend on θ, say $a\theta$ and $b\theta$, where $a < b$ are positive and known. It is but natural to assume that both $a\theta$ and $b\theta$ are increasing functions of θ. Assuming that θ is the only unknown entity, the random variable X has $U(a\theta, b\theta)$ distribution. The problem of interest is to find a point estimate of soil fertility gradient (θ).

In this article, we propose an efficient two-stage procedure for estimating the parameter θ of $U(a\theta, b\theta)$ distribution. Section 2 contains the fixed sample size procedure (FSS) solution and estimation problem. In Sections 3, we propose a two-stage procedure and compute value of $B = B_k$. In Section 4 we give the average sample number (ASN) function and standard deviation (SD) of N_k. In Section 5, some numerical values of ASN function and SD are computed.

2. Fixed Sample Size Procedure

Let $X_1, X_2, \ldots, X_n$ be independent identical distributed (iid) random variables with $U(a\theta, b\theta)$ distribution, where $a < b$ are positive and known. Let $X_{(1)} = \min(X_1, X_2, \ldots, X_n)$ and $X_{(n)} = \max(X_1, X_2, \ldots, X_n)$. Note that $X_{(n)}/b \le \theta \le X_{(1)}/a$ almost surely (as). Then $X_{(n)}/b$ is the maximum likelihood estimator ofθ. That is $\hat{\theta} = X_{(n)}/b$ and the loss function for estimating θ by $\hat{\theta}$ is given by

$$L_n(\hat{\theta},\theta) = A(\hat{\theta}-\theta)^2, \tag{2.1}$$

where A is positive known weight. Our goal is to make the associated risk less than a pre-assigned number w (> 0); that is, $AE\,[(\hat{\theta}-\theta)^2] \le w$.

The risk in estimating θ by $\hat{\theta}$ is $R(\hat{\theta},\theta) = AE[(\hat{\theta}-\theta)^2] = 2A\theta^2(b-a)^2/(n+1)(n+2)b^2$ and this risk will be at most w that is $R(\hat{\theta},\theta) \le w$, which implies $2A\theta^2(b-a)^2/(n+1)(n+2)b^2 \le w$. We know that $(n+1)(n+2) > (n+1)^2$. So, we have $(n+1) \ge \sqrt{\dfrac{2A\theta^2(b-a)^2}{wb^2}}$ that is $n \ge \sqrt{\dfrac{2A\theta^2(b-a)^2}{wb^2}} - 1 = n^*$, where n^* is called the "optimal fixed sample size". When θ is unknown, FSS procedure fails. In the light of this problem, we propose an efficient two-stage procedure.

3. Two-stage Procedure

Stage 1: For a fixed k, take an initial sample $X_1, X_2, \ldots, X_k$ from $U(a\theta, b\theta)$ distribution. Then determine $D = \min(X_1, X_2, \ldots, X_k)$ and $X = \max(X_1, X_2, \ldots, X_k)$. Take $\hat{\theta} = X/b$. We propose the stopping rule:

$$N_k = N(k,B,w) = \max\left\{ k, \ \left\lfloor \sqrt{2BX^2(b-a)^2/wb^4} \right\rfloor + 1 \right\}, \tag{3.1}$$

where B is a positive coefficient and $\lfloor x \rfloor$ denotes the largest integer less than$x$. The coefficient B will be determined appropriately as the risk is bounded by w. We will see that B is only a function of A, k, a and b. While B is known, if $N_k = k$, stop and do not take more observation in the second stage, otherwise go to the second stage.

Stage 2: If $N_k > k$, the initial sample is not large enough, we must gather $N_k - k$ additional observation in the second stage, say $X_{k+1}, X_{k+2}, \ldots, X_{N_k}$. Let $Z = \max(X_{k+1}, X_{k+2}, \ldots, X_{N_k})$. We estimate the parameter θ by $\hat{\theta}_{N_k} = Y^* = \max(X,Z)/b$. The risk associated with this estimator is given by $AE[(\hat{\theta}_{N_k} - \theta)^2]$. If F_k is the σ-field generated by $X_1, X_2, \ldots X_k$ then $X_{k+1}, X_{k+2}, \ldots$..are independent of F_k. Now we obtain the value of B. Now, we obtain the value of B:

$$R(\hat{\theta}_{N_k},\theta) = AE\,[(\hat{\theta}_{N_k}-\theta)^2] = AE\,[(Y^*-\theta)^2] = AE\left\{E\left[\left(\frac{\max(X,Z)}{b}-\theta\right)^2 |F_k\right]\right\}$$

$$= AE\left\{E\left[(X/b-\theta)^2 |F_k\right]\right\} + AE\left\{E\left[(Z/b-\theta)^2 |F_k\right]\right\}$$

$$= \frac{A}{b^2}E\left\{E\left[(X-b\theta)^2 |F_k\right]\right\} + \frac{A}{b^2}E\left\{E\left[(Z-b\theta)^2 |F_k\right]\right\}.$$

We know that there are k samples in the first stage and $(N_k - k)$ samples in the second stage. Thus

$$R(\hat{\theta}_{N_k},\theta) = \frac{A}{b^2}E\left\{\frac{N_k-k+k}{N_k}E\left[(X-b\theta)^2 |F_k\right]\right\} + \frac{A}{b^2}E\left\{\frac{N_k-k+k}{N_k}E\left[(Z-b\theta)^2 |F_k\right]\right\}.$$

Since

$$kE[(X-b\theta)^2] < (N_k-k+k)E[(X-b\theta)^2]$$

and

$$(N_k-k)E[(Z-b\theta)^2] < (N_k-k+k)E[(Z-b\theta)^2],$$

so we have

$$R(\hat{\theta}_{N_k},\theta) > \frac{A}{b^2}E\left\{\frac{k}{N_k}E\left[(X-b\theta)^2 |F_k\right]\right\} + \frac{A}{b^2}E\left\{\frac{N_k-k}{N_k}E\left[(Z-b\theta)^2 |F_k\right]\right\}.$$

We can write

$$\frac{kE[(X-b\theta)^2]}{N_k^{\ 2}} < \frac{kE[(X-b\theta)^2]}{N_k} \text{ and } \frac{(N_k-k)}{N_k^{\ 2}}E[(Z-b\theta)^2] < \frac{(N_k-k)}{N_k}E[(Z-b\theta)^2].$$

$$R(\hat{\theta}_{N_k},\theta) > \frac{A}{b^2}E\left\{\frac{k}{N_k^{\ 2}}E\left[(X-b\theta)^2 |F_k\right]\right\} + \frac{A}{b^2}E\left\{\frac{N_k-k}{N_k^{\ 2}}E\left[(Z-b\theta)^2 |F_k\right]\right\}$$

$$R(\hat{\theta}_{N_k},\theta) > \frac{A}{b^2}E\left\{\frac{k}{N_k^{\ 2}}E\left[(X-b\theta)^2 |F_k\right]\right\} + \frac{A}{b^2}E\left\{\frac{N_k-k}{N_k^{\ 2}}E\left[\left(Z-\left\langle b\theta-\frac{(b-a)\theta}{N_k-k+1}+\frac{(b-a)\theta}{N_k-k+1}\right\rangle\right)^2 |F_k\right]\right\}$$

$$R(\hat{\theta}_{N_k},\theta) > \frac{A}{b^2}E\left\{\frac{k}{N_k^{\,2}}E\left[\left(X-b\theta\right)^2|F_k\right]\right\}+\frac{A}{b^2}E\left\{\frac{N_k-k}{N_k^{\,2}}E\left[\left(Z-\left\langle b\theta-\frac{(b-a)\theta}{N_k-k+1}\right\rangle+\frac{(b-a)\theta}{N_k-k+1}\right)^2|F_k\right]\right\}$$

$$R(\hat{\theta}_{N_k},\theta) > \frac{A}{b^2}E\left\{\frac{k}{N_k^{\,2}}E\left[\left(X-b\theta\right)^2|F_k\right]\right\}+\frac{A}{b^2}E\left\{\frac{N_k-k}{N_k^{\,2}}E\left[\left(Z-\left\langle b\theta-\frac{(b-a)\theta}{N_k-k+1}\right\rangle\right)^2+\frac{(b-a)^2\theta^2}{(N_k-k+1)^2}|F_k\right]\right\}.$$

Since algebraic sum of deviation of observations about its mean is zero, we have

$$R(\hat{\theta}_{N_k},\theta) > \frac{A}{b^2}E\left\{\frac{k}{N_k^{\,2}}\left(X-b\theta\right)^2\right\}+\frac{A}{b^2}E\left\{\frac{N_k-k}{N_k^{\,2}}\left[\mathrm{var}(Z)+\frac{(b-a)^2\theta^2}{(N_k-k+1)^2}\right]\right\}$$

$$R(\hat{\theta}_{N_k},\theta) > \frac{A}{b^2}E\left\{\frac{k}{N_k^{\,2}}\left(X-b\theta\right)^2\right\}+\frac{A}{b^2}E\left\{\frac{N_k-k}{N_k^{\,2}}\left[\frac{(N_k-k)(b-a)^2\theta^2}{(N_k-k+1)^2(N_k-k+2)}+\frac{(b-a)^2\theta^2}{(N_k-k+1)^2}\right]\right\}.$$

$$R(\hat{\theta}_{N_k},\theta) > \frac{A}{b^2}\left(E\left\{\frac{k}{N_k^{\,2}}\left(X-b\theta\right)^2\right\}+E\left\{\frac{(N_k-k)(b-a)^2\theta^2}{N_k^{\,2}(N_k-k+1)^2}\left[\frac{N_k-k}{(N_k-k+2)}+1\right]\right\}\right). \quad (3.2)$$

$$R(\hat{\theta}_{N_k},\theta) > \frac{A}{b^2}\left(J_1+J_2\right),$$

where $J_1 = E\left\{\frac{k}{N_k^{\,2}}\left(X-b\theta\right)^2\right\}$

and $J_2 = E\left\{\frac{(N_k-k)(b-a)^2\theta^2}{N_k^{\,2}(N_k-k+1)^2}\left[\frac{N_k-k}{(N_k-k+2)}+1\right]\right\}.$

Now, $J_1 = E\left\{\frac{k}{N_k^{\,2}}\left(X-b\theta\right)^2\right\}$

$$= E\left\{\frac{k}{N_k^{\,2}}\left(X^2-2Xb\theta+b^2\theta^2\right)\right\} = E\left\{\frac{kb^4w}{2BX^2(b-a)^2}\left(X^2-2Xb\theta+b^2\theta^2\right)\right\}$$

$$J_1 = E\left\{\frac{kb^4w}{2B(b-a)^2}\left(1-\frac{2b\theta}{X}+\frac{b^2\theta^2}{X^2}\right)\right\} \quad (3.3)$$

and $J_2 = E\left\{\frac{(N_k-k)(b-a)^2\theta^2}{N_k^{\,2}(N_k-k+1)^2}\left[\frac{N_k-k}{(N_k-k+2)}+1\right]\right\}$

$$J_2 = E\left\{\frac{(N_k-k)}{N_k^{\,2}(N_k-k+1)}\frac{2(b-a)^2\theta^2}{(N_k-k+2)}\right\}.$$

We know $(N_k - k +1)\,(N_k - k + 2) > (N_k - k)^2$ and $1/(N_k - k)^2 < 1/(N_k - k)$. Further

$$J_2 < E\left\{\frac{(N_k - k)}{N_k^{\ 2}}\frac{2(b-a)^2\theta^2}{(N_k - k)}\right\} = E\left\{\frac{2(b-a)^2\theta^2}{N_k^{\ 2}}\right\},$$

$$= E\left\{\frac{2(b-a)^2\theta^2 wb^4}{2BX^2(b-a)^2}\right\} = \frac{wb^4}{B}E\left\{\frac{\theta}{X}\right\}^2. \tag{3.4}$$

Let $Y = \max(Y_1, Y_2, \ldots\ldots.Y_k) = \max(X_1, X_2, \ldots., X_k)/\theta = X/\theta$, where $Y_i \to U(a,b)$. Therefore, $g(Y) = \dfrac{k(Y-a)^{k-1}}{(b-a)^k}$ so that Equations (3.3) and (3.4) become

$$J_1 = \frac{kb^4 w}{2B(b-a)^2}\left(1 - 2bE\left\langle\frac{\theta}{X}\right\rangle + b^2 E\left\langle\frac{\theta^2}{X^2}\right\rangle\right)$$

$$= \frac{kb^4 w}{2B(b-a)^2}\left(1 - \frac{2b(k+1)}{(bk+a)} + \frac{b^2(k+1)(k+2)}{(b^2k^2 + b^2k + 2abk + 2a^2)}\right)$$

$$= \frac{kb^4 w}{2B(b-a)^2}\left(\frac{2a(b-a)^2}{(bk+a)(b^2k^2 + b^2k + 2abk + 2a^2)}\right)$$

$$J_1 = \frac{akb^4 w}{B(bk+a)(b^2k^2 + b^2k + 2abk + 2a^2)}, \tag{3.5}$$

$$J_2 = \frac{wb^4}{B}E\left\{\frac{\theta}{X}\right\}^2$$

$$= \frac{wb^4(k+1)(k+2)}{B(b^2k^2 + b^2k + 2abk + 2a^2)}. \tag{3.6}$$

Taking addition of Equations (3.5) and (3.6), we get the lower bound for risk as below

$$R(\hat{\theta}_{N_k}, \theta) > \frac{A}{b^2}\left(J_1 + J_2\right) = \frac{Awb^2\left\{k^3b + k^2(3b+a) + k(2b+4a) + 2a\right\}}{B(bk+a)(b^2k^2 + b^2k + 2abk + 2a^2)}.$$

But $R(\hat{\theta}_{N_k}, \theta) \le w$, it is sufficient that

$$B = Ab^2\frac{\left\{k^3b + k^2(3b+a) + k(2b+4a) + 2a\right\}}{(bk+a)(b^2k^2 + b^2k + 2abk + 2a^2)}. \tag{3.7}$$

4. Distribution of *Nk*

The random variable N_k is defined by (3.1), it can take the values $\{k,\ k+1, \ldots\}$ and hence it is discrete random variable.

$$\text{Let } \lambda_j = \frac{jb^2}{\theta(b-a)}\sqrt{\frac{w}{2B}} \tag{4.1}$$

$$P(N_k < \infty) = \sum_{n=k}^{\infty} P(N_k = n)$$

$$= P(N_k = k) + \sum_{n=k+1}^{\infty} P(N_k = n)$$

$$= P\left(\sqrt{\frac{2BX^2(b-a)^2}{wb^4}} \le k\right) + \sum_{n=k+1}^{\infty} P\left(n-1 < \sqrt{\frac{2BX^2(b-a)^2}{wb^4}} < n\right)$$

$$= P\left(\frac{X}{\theta} \le \frac{k}{\theta}\sqrt{\frac{wb^4}{2B(b-a)^2}}\right) + \sum_{n=k+1}^{\infty} P\left(\frac{(n-1)}{\theta}\sqrt{\frac{wb^4}{2B(b-a)^2}} < \frac{X}{\theta} < \frac{n}{\theta}\sqrt{\frac{wb^4}{2B(b-a)^2}}\right)$$

$$= P\left(\frac{X}{\theta} \le \lambda_k\right) + \sum_{n=k+1}^{\infty} P\left(\lambda_{n-1} < \frac{X}{\theta} < \lambda_n\right)$$

$$= P(Y \le \lambda_k) + \sum_{n=k+1}^{\infty} P(\lambda_{n-1} < Y < \lambda_n) \text{ where } Y = X/\theta.$$

$$= \int_{-\infty}^{\lambda_k} \frac{k}{(b-a)^k}(y-a)^{k-1}\mathrm{d}y + \sum_{n=k+1}^{\infty} \int_{\lambda_{n-1}}^{\lambda_n} \frac{k}{(b-a)^k}(y-a)^{k-1}\mathrm{d}y$$

$$= \int_{-\infty}^{\infty} \frac{k}{(b-a)^k}(y-a)^{k-1}\mathrm{d}y = 1\,. \tag{4.2}$$

Thus, the stopping rule is closed.

Now, we develop the formulas of the first and second moments of N_k.

$$E(N_k) = k + \sum_{j=1}^{\infty} P(N_k \ge k+j) = k + \sum_{j=1}^{\infty} P\left(\sqrt{\frac{2BX^2(b-a)^2}{wb^4}} \ge k+j-1\right)$$

$$= k + \sum_{j=0}^{\infty} P\left(\frac{X}{\theta} > \frac{(k+j)}{\theta}\sqrt{\frac{wb^4}{2B(b-a)^2}}\right)$$

$$= k + \sum_{j=0}^{\infty} P\left(Y > \frac{(k+j)b^2}{\theta(b-a)}\sqrt{\frac{w}{2B}}\right), \text{ where } Y = X/\theta \text{ and } \lambda_{k+j} = \frac{(k+j)b^2}{\theta(b-a)}\sqrt{\frac{w}{2B}}.$$

$$= k + \sum_{j=0}^{\infty} P(Y > \lambda_{k+j})$$

$$= k + \sum_{j=0}^{\infty} (1 - P(Y \le \lambda_{k+j}))$$

$$E(N_k) = k + \sum_{j=0}^{\infty} (1 - F_Y(\lambda_{k+j})), \tag{4.3}$$

and $E(N_k^2) = \sum_{n=k}^{\infty} n^2 P(N_k = n)$

$$= k^2 + 2k\sum_{j=1}^{\infty} P(N_k \ge k+j) + \sum_{j=1}^{\infty} j^2 P(N_k = k+j)$$

$$= k^2 + 2k\sum_{j=1}^{\infty} P\left(\sqrt{\frac{2BX^2(b-a)^2}{wb^4}} > k+j-1\right) + \sum_{j=1}^{\infty} j^2 P\left(k+j-1 < \sqrt{\frac{2BX^2(b-a)^2}{wb^4}} < k+j\right)$$

$$= k^2 + 2k\sum_{j=1}^{\infty} P\left(\frac{X}{\theta} > \frac{(k+j-1)b^2}{\theta(b-a)}\sqrt{\frac{w}{2B}}\right)$$

$$+\sum_{j=1}^{\infty} j^2 P\left(\frac{(k+j-1)b^2}{\theta(b-a)}\sqrt{\frac{w}{2B}} < \frac{X}{\theta} < \frac{(k+j)b^2}{\theta(b-a)}\sqrt{\frac{w}{2B}}\right)$$

$$= k^2 + \sum_{j=1}^{\infty}\left\{2kP(Y > \lambda_{k+j-1}) + j^2 P(\lambda_{k+j-1} < Y < \lambda_{k+j})\right\},$$

where $Y = X/\theta$ and

$$\lambda_{k+j} = \frac{(k+j)b^2}{\theta(b-a)}\sqrt{\frac{w}{2B}}, \quad \lambda_{k+j-1} = \frac{(k+j-1)b^2}{\theta(b-a)}\sqrt{\frac{w}{2B}}$$

$$E(N_k^2) = k^2 + \sum_{j=1}^{\infty}\left\{2k(1 - F_Y(\lambda_{k+j-1})) + j^2(F_Y(\lambda_{k+j}) - F_Y(\lambda_{k+j-1}))\right\}, \tag{4.4}$$

where $F_Y(\lambda_{k+j})$ is cumulative distribution function (cdf) of Y and

$F_Y(\lambda_{k+j}) = P(Y \le \lambda_{k+j}) = \dfrac{(\lambda_{k+j} - a)^k}{(b-a)^k}$; for $a \le \lambda_{k+j} \le b$.

Hence, the variance of N_k is

$$V(N_k) = E(N_k^2) - (E(N_k))^2. \tag{4.5}$$

5. Simulation Results

In this section, we compute optimal fixed sample size (n^*) ASN and SD by simulation based on 10,000 repetitions. We take $A = 2$, $\theta = 15$ and $\theta = 10$ and $w = 1, 0.5, 0.25, 0.1, 0.05, 0.025, 0.01$. Pseudorandom samples from uniform population are drawn by using *R* programme. We compute simulated risk ($\hat{R}$) = $E[(\hat{\theta}_{N_k} - \theta)^2]$ (see Tables 1–6).

Remark 5.1: From Tables 1–6, we observe that as value of w decreases, n^*, E(N_k) and SD increases.

Remark 5.2: From Tables 1–6, we observe that as the value of k increases, SD decreases and E(N_k), first, increases, then slightly decreases.

Remark 5.3: From Tables 1–6, we observe that as the value of θ increases, E(N_k) and SD increases.

Remark 5.4: From Tables 1–6, we observe that as value of parameter b increases, E(N_k) and SD increases.

Remark 5.5: From Tables 1–6, we observe that the simulated risk is much less than the pre-assigned number w. Hence, one can adjust the coefficient B such that the risk remains less than w.

Table 1. Numerical values of ASN and SD of rule (3.1)

w	1	0.5	0.25	0.1	0.05	0.025	0.01
n^*	9	13.1421	19	30.6228	43.7214	62.2456	99
$E(N_k)$	10.6213	14.6263	20.4504	32.1421	45.2439	63.8077	100.5816
SD	0.4851	0.6238	0.8408	1.4043	1.9472	2.7767	4.3552
$\widehat{R}$	0.3187	0.1818	0.1078	0.0451	0.0225	0.0119	0.0048

$k = 30, A = 2, \theta = 10, b = 2, a = 1$

w	1	0.5	0.25	0.1	0.05	0.025	0.01
n^*	9	13.1421	19	30.6228	43.7214	62.2456	99
$E(N_k)$	30	30	30	32.1157	45.2811	63.7773	100.5279
SD	0	0	0	0.6120	0.8129	1.0791	1.6248
$\widehat{R}$	0.0488	0.0507	0.0475	0.0443	0.0234	0.0120	0.0048

Source: All ta Table is obtained by using rule (3.1). Created by author.

Table 2. Numerical values of ASN and SD of rule (3.1)

w	1	0.5	0.25	0.1	0.05	0.025	0.01
n^*	14	20.2132	29	46.4342	66.0820	93.8683	149
$E(N_k)$	15.5218	21.7447	30.5520	47.9793	67.64801	95.4580	150.6308
SD	0.6956	0.9946	1.3651	2.0857	2.9431	4.1467	6.5352
$\widehat{R}$	0.3696	0.2027	0.1057	0.0462	0.0243	0.0121	0.0050

$k = 30, A = 2, \theta = 15, \ b = 2, a = 1$

w	1	0.5	0.25	0.1	0.05	0.025	0.01
n^*	14	20.2132	29	46.4342	66.08204	93.8683	149
$E(N_k)$	30	30	30.6288	47.9541	67.5907	95.4052	150.5347
SD	0	0	0.4831	0.8431	1.1168	1.5600	2.4154
$\widehat{R}$	0.1169	0.1141	0.1076	0.0458	0.0229	0.0124	0.0049

Source: Table is obtained by using rule (3.1). Created by author.

Table 3. Numerical values of ASN and SD of rule (3.1)

w	1	0.5	0.25	0.1	0.05	0.025	0.01
n^*	14	20.2132	29	46.4342	66.0820	93.8683	149
$E(N_k)$	15.4444	21.6913	30.4545	47.8678	67.4877	95.2481	150.3052
SD	1.0112	1.4442	2.0182	3.1712	4.4760	6.3384	10.0108
$\widehat{R}$	0.4056	0.2030	0.1128	0.0456	0.0242	0.0119	0.0048

$k = 30, A = 2, \theta = 10, b = 4, a = 1$

w	1	0.5	0.25	0.1	0.05	0.025	0.01
n^*	14	20.2132	29	46.4342	66.0820	93.8683	149
$E(N_k)$	30	30	30.6236	47.9533	67.5769	95.3534	150.481
SD	0	0	0.4845	1.1938	1.6362	2.2987	3.6165
$\widehat{R}$	0.1155	0.1140	0.1047	0.0465	0.0224	0.0118	0.0049

Source: Table is obtained by using rule (3.1). Created by author.

Table 4. Numerical values of ASN and SD of rule (3.1)

w	1	0.5	0.25	0.1	0.05	0.025	0.01
n^*	21.5	30.8198	44	70.1512	99.6231	141.3025	224
$E(N_k)$	22.9546	32.2648	45.4434	71.5610	100.9897	142.6208	225.206
SD	1.5162	2.1298	3.0193	4.7583	6.7158	9.5011	15.0110
$\widehat{R}$	0.4320	0.2215	0.1167	0.0485	0.0245	0.0124	0.0051

$k = 30,\ A = 2, \theta = 15, b = 4, a = 1.$

w	1	0.5	0.25	0.1	0.05	0.025	0.01
n^*	21.5	30.8198	44	70.1512	99.6231	141.3025	224
$E(N_k)$	30	32.3626	45.4673	71.6163	101.0929	142.7772	225.4703
SD	0	0.7932	1.1037	1.7047	2.4203	3.4244	5.4153
$\widehat{R}$	0.2609	0.2204	0.1175	0.0477	0.0242	0.0120	0.0050

Source: Table is obtained by using rule (3.1). Created by author.

Table 5. Numerical values of ASN and SD of rule (3.1)

w	1	0.5	0.25	0.1	0.05	0.025	0.01
n^*	9	13.1421	19	30.6228	43.7214	62.24555	99
$E(N_k)$	10.6213	14.6262	20.4504	32.1421	45.2439	63.8076	100.5816
SD	0.4850	0.6238	0.8408	1.4043	1.9472	2.7767	4.3552
$\widehat{R}$	0.3114	0.1795	0.1085	0.0446	0.0228	0.0115	0.0048

$k = 30, A = 2, \theta = 10, b = 4, a = 2.$

w	1	0.5	0.25	0.1	0.05	0.025	0.01
n^*	9	13.1421	19	30.6228	43.7214	62.2455	99
$E(N_k)$	30	30	30	32.1157	45.2810	63.7773	100.5279
SD	0	0	0	0.6120	0.8128	1.0791	1.6248
$\widehat{R}$	0.0506	0.0509	0.0511	0.0447	0.0229	0.0116	0.0049

Source: Table is obtained by using rule (3.1). Created by author.

Table 6. Numerical values of ASN and SD of rule (3.1)

w	1	0.5	0.25	0.1	0.05	0.025	0.01
n^*	14	20.2132	29	46.4342	66.0820	93.8683	149
$E(N_k)$	15.5218	21.7447	30.5520	47.9793	67.6480	95.4580	150.6308
SD	0.6956	0.9946	1.3651	2.0857	2.9431	4.1467	6.5352
$\widehat{R}$	0.3626	0.2005	0.1063	0.0459	0.0242	0.0120	0.0048

$k = 30, A = 2, \theta = 15, b = 4, a = 2$.

w	1	0.5	0.25	0.1	0.05	0.025	0.01
n^*	14	20.2132	29	46.4341	66.0820	93.86833	149
$E(N_k)$	30	30	30.6288	47.9541	67.5907	95.4052	150.5347
SD	0	0	0.48311	0.8431	1.1168	1.5600	2.4154
$\widehat{R}$	0.1129	0.1113	0.1077	0.0452	0.0235	0.0123	0.0049

Source: Table is obtained by using rule (3.1). Created by author.

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References

1. Graybill A, Connell TL. Sample size required to estimate the parameter in uniform density within 'd' units of true value. *J Am Stat Assoc*. 1964; 59: 550–556.
2. Cooke PJ. Sequential estimation in the uniform density. *J Am Stat Assoc*. 1971; 66(335): 614–617.
3. Cooke PJ. Two-stage sequential estimation in uniform density. *J Am Stat Assoc*. 1973; 68(341): 107–108.
4. Govindarajulu Z. A note no two-stage and sequential fixed-width intervals for the parameter in the uniform density. *Stat Probab Lett*. 1997; 36: 179–188.
5. Govindarajulu Z. Erratum: a note on two-stage and sequential fixed-width intervals for the parameter in the uniform density. *Stat Probab Lett*. 1999; 42: 213–215.
6. Akahira M, Koike K. Sequential interval estimation of a location parameter with the fixed width in the uniform distribution with an unknown scale parameter. *Seq Anal*. 2005; 24: 63–75.
7. Koike K. Sequential interval estimation of a location parameter with fixed width in the non-regular case. *Seq Anal*. 2007; 26: 63–70.
8. Patil HS, Rattihalli RN. Fixed width confidence interval for $U(\theta, m\theta)$ distribution. *Commun Stat* (*Theory Methods)*. 2011; 40: 907–915.
9. Bhattacharjee D, Mukhopadhyay N. On recovering lost information for sequential estimation in a uniform distribution. Proceedings of The 58th World Statistical Congress (Session STS005). International Statistical Institute, Dublin, Ireland; 2011.

10. Patil HS. Two-stage estimation procedure for the parameter of $U(\boldsymbol{\theta}, m\theta)$ distribution. *Calcutta Stat Assoc Bull.* 2012; 64: 255–256, 257–263.
11. Bhattacharjee D, Mukhopadhyay N. On sequential point estimation in a uniform distribution with adjusted non-sufficient estimators: a comparative study and data illustration. *Calcutta Stat Assoc Bull.* 2013; 65: 103–122.
12. Patil HS. A purely sequential estimation procedure for the parameter of $U(\boldsymbol{\theta}, m\theta)$ distribution. *Calcutta Stat Assoc Bull*. 2014; 66: , 235–240, 263–264.
13. Zacks S, Mukhopadhyay N. Bounded risk estimation of the exponential parameter in a two-stage sampling. *Seq Anal*. 2006a; 25: 437–452.
14. Zacks S, Mukhopadhyay N. Exact risk of sequential point estimators of the exponential parameter. *Seq Anal*. 2006b; 25: 203–220.
15. Mukhopadhyay N, Pepe W. Exact bounded risk estimation when the terminal sample size and estimator are dependent: The exponential cases. *Seq Anal*. 2006; 29: 85–101.
16. Zacks S. The exact distributions of the stopping times and their functionals in two-stage and sequential fixed-width confidence intervals of the exponential parameter. *Seq Anal*. 2009; 28: 69–81
17. Zacks S, Khan A. Two-stage and sequential estimation of the scale parameter of a gamma distribution with fixed-width intervals. *Seq Anal*. 2011; 30: 297–307.
18. Mahmoudi E, Roughani G. Bounded risk estimation of the scale parameter of a gamma distribution in a two-stage sampling procedure. *Seq Anal*. 2015; 34: 25–38.
19. Ghosh M, Mukhopadhyay N, Sen PK. *Sequential Estimation.* New York, NY: Wiley.